Announcements

1) HW 2 supplement due today

2) Quiz Thursday, Exam 1 next week

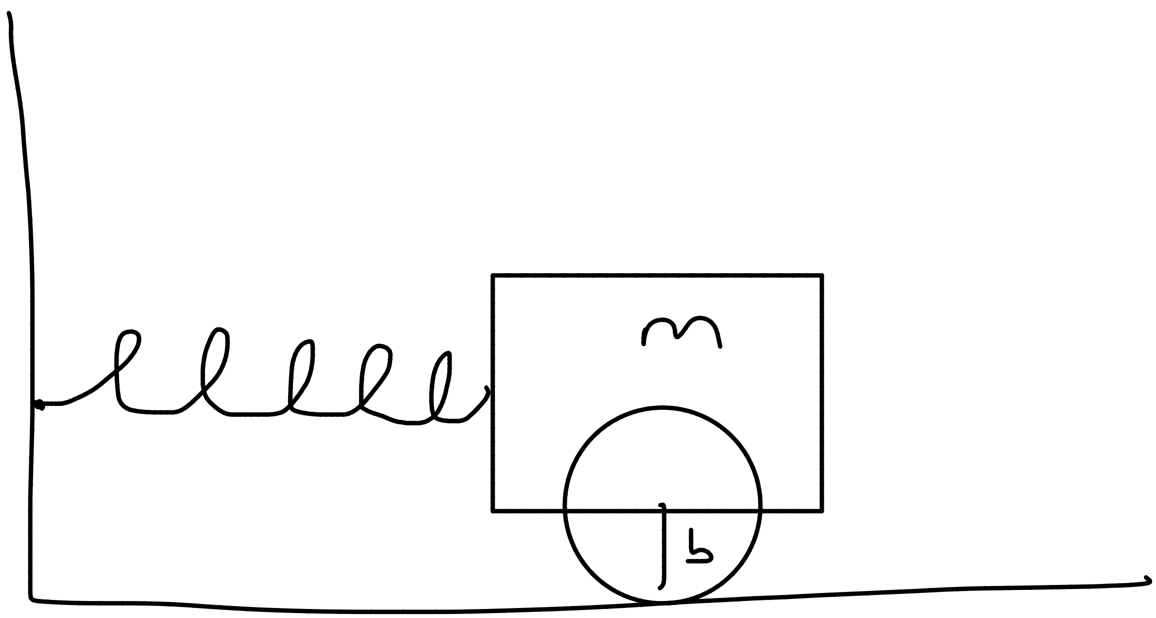
Second Order Equations

(Chapter 2)

Application: (mass-spring oscillator)

A mass m (in kg, say) is attached to a spring and equipped with a "wheel" of radius b.

Picture



Hooke's Law

$$F_{spring}(t) = -k\,t$$

$$(t = \text{displacement})$$

Not sophisticated enough to deal with large displacements, so we combine Newton's 2^{nd} law ($F = ma$) with a friction law

$$F_{friction} = -bv$$

where v is the velocity and $b \geq 0$ is a damping coefficient.

Carelessly lump all other forces into an "external" force function. Let y be the displacement as a function of time.

By Newton's law,

$$\boxed{\begin{aligned} m\frac{d^2y}{dt^2} &= F_{spring}(t) + F_{friction}(t) + F_{external}(t) \\ &= -ky - b\frac{dy}{dt} + F_{external}(t) \end{aligned}}$$

We get, replacing

$\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ with y', y'',

respectively,

$$\boxed{F_{ext}(t) = my'' + by' + ky}$$

Difference between Second-order and First-order Differential Equations

1) Initial conditions

In first-order equations, you need to specify $y(t_0) = y_0$ in order to solve for y completely. We also need to specify $y'(t_1) = a$ for second order equations

2) Method of solution:

we don't know any!

Aside from simple differential equations that you can reduce to first order equations (i.e. $k=0$ in our formula), we have no way of solving these equations.

Example 1: (some numbers)

Suppose in the mass-spring oscillator, $F_{ext}(t) = 0$, $m = 1$ kg, $k = 25$, and $b = 10$. Can we solve for y?

Equation:

$$0 = y'' + 10y' + 25y$$

This might make you think about quadratic equations.

But how to replace the primes with squares, first powers, etc.

Guess: $y = e^{rt}$ for some real number r.

Check our guess!

Plug into the equation:

$$y(t) = e^{rt}, \quad y'(t) = re^{rt},$$

$$y''(t) = r^2e^{rt}$$

chain rule

Substituting,

$$0 = r^2e^{rt} + 10re^{rt} + 25e^{rt}$$

$$= e^{rt}(r^2 + 10r + 25)$$

$e^{rt} > 0$, so divide.

We get

$$0 = r^2 + 10r + 25$$
$$= (r+5)^2,$$

so $r = -5$, and

$\boxed{y = e^{-5t}}$ is

a solution to our problem.

Observation: (analogy to linear algebra)

If y_1 and y_2 are solutions to

$$0 = y'' + ay' + by,$$ then

1) cy_1 is also a solution

(in particular, $c=0$)

2) $y_1 + y_2$ is also a solution

This says that for any constants c_1, c_2 and solutions y_1, y_2,

$$y = c_1 y_1 + c_2 y_2$$ is

also a solution. This is because our differential equation is linear.

More guessing: We should expect, since we take two derivatives, two "honestly" different solutions. But we only got e^{-5t} (and constant multiples) as a solution to our equation: where is the other solution?

Another guess:

$$y = t e^{-5t}.$$

Check: equation

$$0 = y'' + 10y' + 25y.$$

$$y'(t) = e^{-5t} - 5te^{-5t}$$

$$y''(t) = -5e^{-5t} - 5e^{-5t} + 25te^{-5t}$$

$$= -10e^{-5t} + 25te^{-5t}$$

(product rule)

Plugging in, we get

$$-10e^{-5t} + 25te^{-5t} +$$

$$10\left(-5te^{-5t} + e^{-5t}\right)$$

$$+ 25\left(te^{-5t}\right)$$ $\longrightarrow$ pull out e^{-5t},

leaving

$$-\cancel{10} + \cancel{25t} - \cancel{50t} + \cancel{10}$$

$$+\cancel{25t} = 0$$

We now have two "honestly" different solutions,

$$y_1(t) = e^{-5t} \text{ and}$$

$$y_2(t) = te^{-5t}.$$

What do we mean by "honestly"?

Q: How do we know these are all the solutions?

Linear Dependence: Two functions f and g, defined on an interval I, are said to be linearly dependent if there is a real number c such that

$$\boxed{f(x) = cg(x)}$$

for all x in I.

Two functions that are not linearly dependent are called linearly independent.

$y_1(t) = e^{-5t}$ and

$y_2(t) = te^{-5t}$ are

linearly independent!